Complex Analysis: Resit Exam

Aletta Jacobshal 01, Friday 12 April 2019, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** on the envelope and at the top of each answer sheet.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope). The exam and its solutions will be uploaded to Nestor in the following days.
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.

Question 1 (10 points)

Consider a function $f(z)$ such that $\text{Re}(f(z)) \geq M$ for all $z \in \mathbb{C}$, where M is a real constant. Prove that if $f(z)$ is entire then it must be constant. Hint: consider the function $e^{-f(z)}$.

Solution

Let

$$g(z) = e^{-f(z)}.$$

If $f(z)$ is entire, then so is $g(z)$. Moreover, if we write $f = u + iv$, with $u = \text{Re}(f(z))$, $v = \text{Im}(f(z))$, then we have

$$|g(z)| = |e^{-f(z)}| = |e^{-u-iv}| = |e^{-u}| \leq e^{-M}.$$

Since $g(z)$ is a bounded entire function we conclude from Liouville's theorem that $g(z)$ is constant $c \in \mathbb{C}$.

Therefore, $e^{-f(z)} = c$. This implies $f(z) = -\operatorname{Log} c + 2k(z)\pi i$, with $k(z)$ a $\mathbb{Z}$-valued function. Since $f(z)$ is continuous (being entire) we conclude that $k(z)$ is also continuous. The only continuous functions from $\mathbb{C}$ to $\mathbb{Z}$ are constant functions, therefore $k(z) = K$. Then $f(z) = -\operatorname{Log} c + 2K\pi i$ is a constant function.

Question 2 (20 points)

(a) (8 points) Consider the integral

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{4ix}}{x^2 - 1}\, \mathrm{d}x.$$

Specify and draw a (closed) contour that you can use to compute such an integral with the calculus of residues. Give full justification for your choice of contour. NB: You are *not* being asked to compute this integral.

Solution

There are two issues we have to consider here. First, the factor e^{4ix} in the integral and, second, the fact that the integrand has two poles of order 1 at $x = \pm 1$.

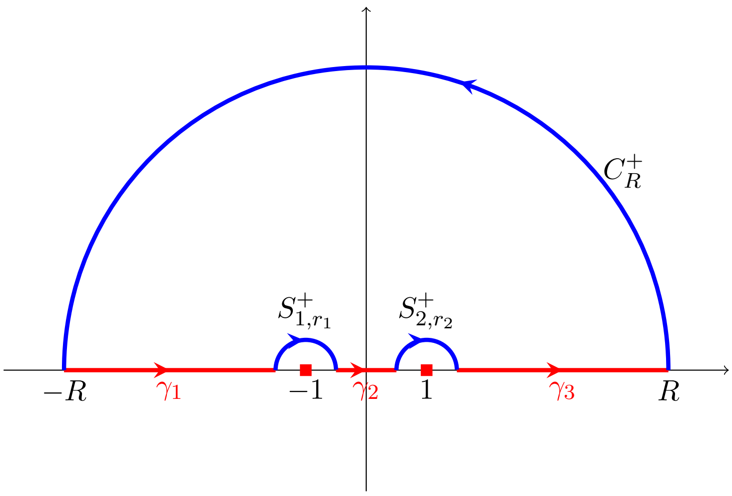
Here we consider the contour

$$\Gamma = \gamma_1 + S^+_{1,r_1} + \gamma_2 + S^+_{2,r_2} + \gamma_3 + C^+_R,$$

where

- γ_1 is the straight line connecting $-R$ to $-1 - r_1$ on the real axis,
- γ_2 is the straight line connecting $-1 + r_1$ to $1 - r_2$,
- γ_3 is the straight line connecting $1 + r_2$ to R,
- S^+_{1,r_1} is the half-circle centered at -1 and connecting $-1 - r_1$ to $-1 + r_1$ in the upper half-plane (although we could have chosen S^-_{1,r_1} in the lower half-plane),
- S^+_{2,r_2} is the half-circle centered at 1 and connecting $1 - r_2$ to $1 + r_2$ in the upper half-plane (although we could have chosen S^-_{2,r_2} in the lower half-plane),
- and C^+_R is the half-circle centered at 0 and connecting R to $-R$ in the upper half-plane.

We need S^+_{1,r_1} and S^+_{2,r_2} (or S^-_{1,r_1} and S^-_{2,r_2}) to bypass the singularities of the integrand at -1 and 1. We need to take C^+_R, instead of C^-_R, so that we can apply Jordan's lemma for e^{4ix} and estimate that the contribution to the integral from C^+_R goes to 0 as $R \to \infty$.



(b) (12 points) Evaluate the integral

$$\text{pv} \int_{-\infty}^{\infty} \frac{x}{(x - i)(x + 2i)(x - 3i)(x + 4i)} \, dx,$$

using the calculus of residues. Give complete arguments.

Solution

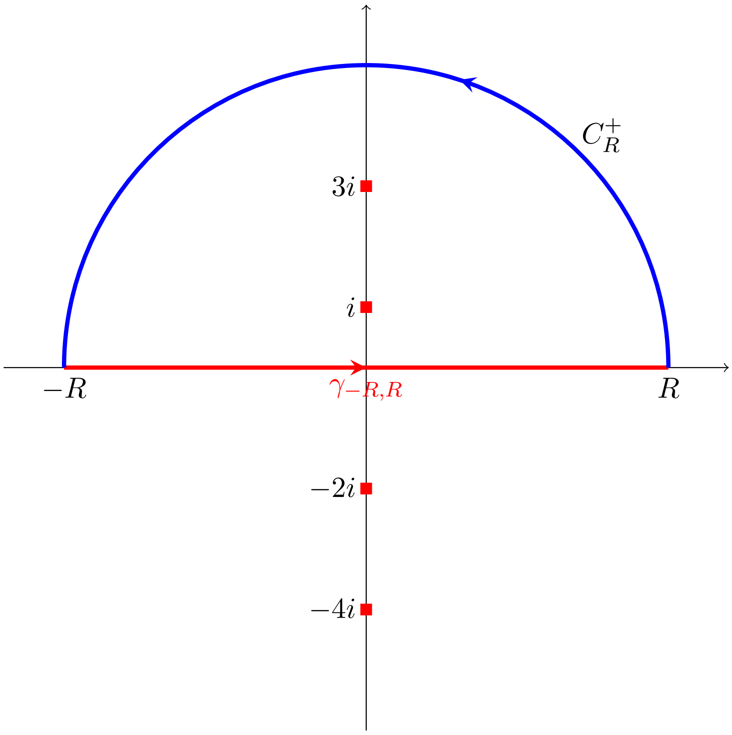
Let

$$I = \text{pv} \int_{-\infty}^{\infty} \frac{x}{(x - i)(x + 2i)(x - 3i)(x + 4i)} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{x}{(x - i)(x + 2i)(x - 3i)(x + 4i)} \, dx.$$

The denominator of the integrand has poles of order 1 at i, −2i, 3i, and −4i. For the integration we consider the contour

$$\Gamma = \gamma_{-R,R} + C_R^+,$$

shown below.



Then we have

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{\gamma_{-R,R}} f(z)\,\mathrm{d}z + \int_{C_R^+} f(z)\,\mathrm{d}z,$$

where

$$f(z) = \frac{z}{(z-\mathrm{i})(z+2\mathrm{i})(z-3\mathrm{i})(z+4\mathrm{i})}.$$

For R large enough ($R > 3$), Γ contains exactly two simple poles i and 3i of $f(z)$. Since i is a simple pole, we have

$$\begin{aligned}
\operatorname{Res}(f,\mathrm{i}) &= \lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{z}{(z-\mathrm{i})(z+2\mathrm{i})(z-3\mathrm{i})(z+4\mathrm{i})} \\
&= \lim_{z\to\mathrm{i}} \frac{z}{(z+2\mathrm{i})(z-3\mathrm{i})(z+4\mathrm{i})} \\
&= \frac{\mathrm{i}}{(3\mathrm{i})(-2\mathrm{i})(5\mathrm{i})} \\
&= \frac{1}{30}.
\end{aligned}$$

Similarly, for the simple pole at 3i we have

$$\begin{aligned}
\operatorname{Res}(f,3\mathrm{i}) &= \lim_{z\to 3\mathrm{i}}(z-3\mathrm{i})\frac{z}{(z-\mathrm{i})(z+2\mathrm{i})(z-3\mathrm{i})(z+4\mathrm{i})} \\
&= \lim_{z\to 3\mathrm{i}} \frac{z}{(z-\mathrm{i})(z+2\mathrm{i})(z+4\mathrm{i})} \\
&= \frac{3\mathrm{i}}{(2\mathrm{i})(5\mathrm{i})(7\mathrm{i})} \\
&= -\frac{3}{70}.
\end{aligned}$$

Therefore, for $R > 3$ we have

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{i}) + 2\pi\mathrm{i}\,\mathrm{Res}(f,3\mathrm{i}) = 2\pi\mathrm{i}\left(\frac{7}{210} - \frac{9}{210}\right) = -\frac{2\pi\mathrm{i}}{105}.$$

Then

$$-\frac{2\pi\mathrm{i}}{105} = \lim_{R\to\infty}\int_{\gamma_{-R,R}} f(z)\,\mathrm{d}z + \lim_{R\to\infty}\int_{C_R^+} f(z)\,\mathrm{d}z.$$

Since $f(z) = P(z)/Q(z)$ with $\deg Q = 4 \geq 3 = \deg P + 2$ we know that the second limit is 0. The first integral is I so we have

$$I = -\frac{2\pi\mathrm{i}}{105}.$$

Question 3 (20 points)

Represent the function

$$f(z) = \frac{z}{z^2-1},$$

(a) (8 points) as a Taylor series around 0 and find its radius of convergence;

Solution

$$\begin{aligned}\frac{z}{z^2-1} &= -z(1 + z^2 + z^4 + z^6 + z^8 + \cdots)\\ &= -z - z^3 - z^5 - z^7 - z^9 \cdots,\end{aligned}$$

where we used the geometric series for $1/(1-z^2)$. The geometric series converges when $|z^2| < 1$, that is, for $|z| < 1$. Therefore, we conclude that the radius of convergence must be 1.

Alternatively, the function $f(z)$ has singularities at $z = \pm 1$ which are both at a distance $|z| = 1$ from 0. Therefore, the radius of convergence is 1.

(b) (7 points) as a Laurent series in the domain $|z| > 1$.

Solution

Since $|z| > 1$, that is $|1/z^2| < 1$, we have

$$\frac{z}{z^2-1} = \frac{\frac{1}{z}}{1-\frac{1}{z^2}} = \frac{1}{z}\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots\right).$$

Therefore, for $|z| > 1$ we can write

$$\frac{z}{z^2-1} = \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \frac{1}{z^7} + \cdots$$

(c) (5 points) Determine the singularities of the function

$$g(z) = \frac{\sin z}{z},$$

and their type (removable, pole, essential; if pole, give the order). Justify your answers.

Solution

The numerator $\sin z$ is entire and the denominator also entire and becomes zero at $z_0 = 0$. Therefore, $g(z)$ has a singularity at $z_0 = 0$.

Since

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{\sin z}{z} = 1,$$

we conclude that the singularity is removable. Alternatively,

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{1}{3!}z^3 + \cdots\right) = 1 - \frac{1}{3!}z^2 + \cdots,$$

leading to the same conclusion (since there are no negative powers in the Laurent series of $g(z)$ for $|z| > 0$.

Question 4 (15 points)

Consider the polynomial $P(z) = z^4 + \varepsilon(z-1)$ where $\varepsilon > 0$. Show that if $\varepsilon < \dfrac{r^4}{1+r}$ then the polynomial P has four zeros inside the circle $|z| = r$.

Solution

The functions $f(z) = z^4$ and $P(z)$ are analytic on and inside the circle $|z| = r > 0$.

The number of zeros of z^4 inside this circle, counting multiplicity, is $N_0(f) = 4$.

Moreover, on the circle $|z| = r$ we have for $h(z) = \varepsilon(z+1)$ that

$$|h(z)| = |\varepsilon(z-1)| \leq \varepsilon(|z|+1) = \varepsilon(r+1) < r^4,$$

and

$$|f(z)| = |z^4| = r^4.$$

Therefore, on the circle $|z| = r$, we find

$$|h(z)| < r^4 = |f(z)|.$$

These facts mean that we can apply Rouché's theorem for $P = f + h$ to get

$$N_0(P) = N_0(f) = 4.$$

Question 5 (15 points)

Compute the following integrals along the path γ shown below that lies in the upper half-plane, starts at -1 and ends at $1 + \mathrm{i}$. Give complete arguments.

(a) (6 points) $\displaystyle\int_\gamma \frac{1}{z^2}\,\mathrm{d}z$.

Solution

We have $(-1/z)' = 1/z^2$. Therefore,

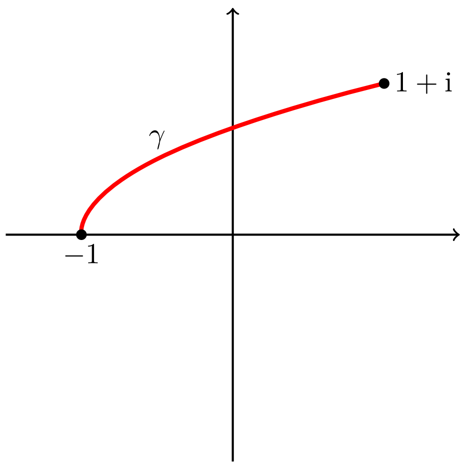
$$\int_\gamma \frac{1}{z^2}\,\mathrm{d}z = -\frac{1}{1+\mathrm{i}} + \frac{1}{-1} = -\frac{1}{2} + \frac{\mathrm{i}}{2} - 1 = -\frac{3}{2} + \frac{\mathrm{i}}{2}.$$

(b) (9 points) $\displaystyle\int_\gamma \frac{1}{z}\,\mathrm{d}z$.

Solution

We have $L'(z) = 1/z$ where $L(z)$ is the branch of the logarithm obtained by taking the argument in the interval $[0, 2\pi)$ (and thus it has a branch cut along the positive real axis). Then

$$\begin{aligned}\int_\gamma \frac{1}{z}\,\mathrm{d}z &= L(1+\mathrm{i}) - L(-1) = \operatorname{Log}|1+\mathrm{i}| + \mathrm{i}A(1+\mathrm{i}) - \operatorname{Log}|-1| - \mathrm{i}A(-1) \\ &= \operatorname{Log}\sqrt{2} + \mathrm{i}\frac{\pi}{4} - \operatorname{Log} 1 - \mathrm{i}\pi = \operatorname{Log}\sqrt{2} - \mathrm{i}\frac{3\pi}{4}.\end{aligned}$$



Question 6 (10 points)

Answer only one of the following two questions:

Question A. Consider the Möbius transformation

$$f(z) = \frac{1+\mathrm{i}}{-\mathrm{i}z + 1}$$

on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. After computing $f(0)$, $f(\pm 1)$, and $f(\pm\mathrm{i})$, and expressing them in standard form (real plus imaginary part), determine the image of the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ under f.

Solution

We first check the images of the given points. We compute

$$f(0) = 1+\mathrm{i},\ \ f(1) = \mathrm{i},\ \ f(-1) = 1,\ \ f(\mathrm{i}) = \tfrac{1}{2}(1+\mathrm{i}),\ \ f(-\mathrm{i}) = \infty.$$

Therefore, since under Möbius transformations circles and lines are mapped to circle and lines, we conclude that the unit circle going through the points ± 1, $\pm\mathrm{i}$ is mapped to the straight line going through the points 1, i, ∞ that is, the line $\ell = \{z \in \mathbb{C} : \operatorname{Im} z = 1 - \operatorname{Re} z\}$.

Moreover, since 0 is mapped to $1+\mathrm{i}$ we conclude that the closed unit disk is mapped to the subset of $\mathbb{C}$ that is 'above' ℓ, that is, to the set $\{z \in \mathbb{C} : \operatorname{Im} z \geq 1 - \operatorname{Re} z\}$.

Question B. Prove that if $f(z)$ is entire and agrees with a polynomial $\sum_{j=0}^{n} a_j x^j$ for $z = x$ on a segment of the real axis, then $f(z) = \sum_{j=0}^{n} a_j z^j$ everywhere.

Solution

This is a direct consequence of Corollary 5 in Section 5.6 on Analytic Continuation.

> **Corollary 5.** If f and g are analytic functions in a domain D and $f(z_n) = g(z_n)$ for an infinite sequence of distinct points $\{z_n\}$ converging to a point z_0 in D, then $f \equiv g$ throughout D.

In particular, since the function $f(z)$ and the function $g(z) = \sum_{j=0}^{n} a_j z^j$ are entire and they agree on a segment of the real axis, we can conclude that $f(z) = g(z)$ throughout $\mathbb{C}$.

Formulas

The Cauchy-Riemann equations for a function $f = u + \mathrm{i}v$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The principal value of the logarithm is

$$\operatorname{Log} z = \operatorname{Log}|z| + \mathrm{i} \operatorname{Arg} z.$$

The generalized Cauchy integral formula is

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

The residue of a function f at a pole z_0 of order m is given by

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z - z_0)^m f(z) \right].$$